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QUANTIZATION OF A MASSIVE SPIN ONE PARTICLE

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Abstract

The Hamilton-Jacobi approach is used to quantize a massive spin one particle, which is a constrained system. The equations of motion for this singular system are obtained as total differential equations in many variables. After writing the equations of motion in a compact notation for vectors, they become equations of the electromagnetic field and Maxwell's equations. The integrability conditions are investigated without using any gauge fixing condition. The canonical method is used to quantize the system. The path integral is obtained as an integration over the canonical phase space coordinates.

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1 Introduction

The quantization of constrained Hamiltonian systems can be achieved by means of operator methods [1,2] or by path integral quantization [3-5].

The canonical method (or Güler's method) was developed the Hamilton-Jacobi formulation to investigate the quantization of constrained systems [6, 7]. The variational principle is the starting point of the Hamilton-Jacobi approach [8-11]. The treatment of constrained systems due to the Hamiltonian leads to obtain the equations of motion as total differential equations in many variables. the equations of motion are integrable when the corresponding system of partial differential equations is a Jacobi system. Güler has presented in Ref. [12] a classically treatment of constrained systems. Afterwards, the quantization of constrained system was investigated by Hamilton-Jacobi quantization as seen in Ref. [13] which is presented the Hamilton-Jacobi quantization of finite dimensional system with constraints. By comparison, Dirac's method with the canonical method (Hamilton-Jacobi approach) in treatment of constrained singular systems, one finds that both give the same results as seen in Refs. [14-18]. But the Hamilton-Jacobi approach is simpler than Dirac's method [19-23] because of: (i) the Hamilton-Jacobi formalism makes no difference between first- and second-class constraints (ii) the gauge fixing is not always an easy task in the case of using Dirac's method, but if the canonical method (Hamilton-Jacobi approach) [7,8] is used, it does not require a gauge-fixing term at all because the gauge variables are separated in the processes of constructing an integrable system of total differential equations.

The quantization of a massive spin one particle became the center of interest of physicists, especially after the pioneering work of Faddeev [3], who introduced the path integral quantization of singular theories that possesses first-class constraints in canonical gauge. Since first class constraints are generators of gauge transformations, which will lead to gauge freedom. In other words, the equations of motion are still degenerate and depending on the functional arbitrariness, one has to impose external gauge constraints for each first-class constraint. Senjanovic [5] generalized Faddeev's method for the quantization and construction of the functional integral for theories with second-class constraints in canonical gauge. However,

based on the canonical method, the path integral formulation obtained in Refs. [23–25]. We investigated the path integral quantization of singular constrained field systems by using the Faddev's method and canonical method in the case of first-class constraints as seen in Ref. [19], and by using the Senjanovic's method and canonical method for the field system which contains first-class constraints and second-class constraints as seen in Ref. [23]. That leads to the same results and shows the simplicity for using the canonical path integral quantization for quantizing the systems.

The purpose of this paper is to use the path integral quantization of systems with finite degrees of freedom in order to include the electromagnetic field with constraints using the canonical method.

This paper is arranged as follows: canonical path integral quantization method is presented in Sec.2. the quantization method is applied to a massive spin one particle in Sec.3. The paper closes with a conclusion in Sec.4.

2 Canonical path integral quantization

In this section, we study the constrained systems by using the canonical method [6,7] and demonstrate the fact that the gauge fixing problem is solved naturally. The starting point of this method is to consider the lagrangian $L \equiv L(q_i, \dot{q}_i, \tau)$, $i = 1, 2, \ldots, n$ with Hess matrix

$$A_{ij} = \frac{\partial^2 L(q_i, \dot{q}_i, \tau)}{\partial \dot{q}_i \, \partial \dot{q}_j}, \qquad i, j = 1, 2, \ldots, n, \tag{1}$$

of rank $(n - r)$, $r < n$. Then r momenta are dependent. The generalized momenta p_i corresponding to the generalized coordinates q_i are defined as

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \qquad a = 1, 2, \ldots, n - r, \tag{2}$$

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \qquad \mu = n - r + 1, \ldots, n. \tag{3}$$

enable us to solve Eq. (2) for $\dot{q}_a$ as

$$\dot{q}_a = \dot{q}_a(q_i, p_a, \dot{q}_\mu; \tau) \equiv \omega_a. \tag{4}$$

Substituting Eq. (4), into Eq. (3), we get

$$p_\mu = \left.\frac{\partial L}{\partial \dot{q}_\mu}\right|_{\dot{q}_a=\omega_a} \equiv -H_\mu(q_i, \dot{q}_\mu, p_a; t). \tag{5}$$

Relations (5) indicate the fact that the generalized momenta p_μ are not independent of p_a which is a natural result of the singular nature of the lagrangian.
The canonical Hamiltonian H_0 is defined as

$$H_0 = -L(q_i, \dot{q}_\mu, \dot{q}_a \equiv \omega_a; \tau) + p_a\dot{q}_a + p_\mu\dot{q}_\mu\big|_{p_\mu=-H_\mu}. \tag{6}$$

The set of Hamilton-Jacobi Partial Differential Equations is expressed as

$$H'_\alpha\left(\tau,\ q_\mu,\ q_a,\ P_i = \frac{\partial S}{\partial q_i},\ P_0 = \frac{\partial S}{\partial \tau}\right) = 0,\ \alpha = 0, n-p+1, \ldots, n, \tag{7}$$

where

$$H'_\alpha = p_\alpha + H_\alpha\ . \tag{8}$$

The equations of motion are obtained as total differential equation in many variables as follows:

$$dq_r = \frac{\partial H'_\alpha}{\partial p_r}\, dt_\alpha, \qquad r = 0, 1, \ldots, n\,, \tag{9}$$

$$dp_a = -\frac{\partial H'_\alpha}{\partial q_a}\, dt_\alpha, \qquad a = 1, \ldots, n-p, \tag{10}$$

$$dp_\mu = -\frac{\partial H'_\alpha}{\partial q_\mu}\, dt_\alpha, \qquad \alpha = 0, n-p+1, \ldots, n, \tag{11}$$

$$dz = \left(-H_\alpha + p_a\frac{\partial H'_\alpha}{\partial p_a}\right) dt_\alpha. \tag{12}$$

where $Z = S(t_\alpha, q_a)$. The set of equations (9-12) are integrable if

$$dH'_\alpha = 0, \qquad \alpha = 0, n-p+1, \ldots, n. \tag{13}$$

If conditions (13) are not satisfied identically, one may consider them as new constraint and a gain test the integrability conditions ,then repeating

this procedure, a set of conditions may be obtained.
In this case of integrable system, the path integral representation may be written as [26-29].

$$\langle Out \mid S \mid In \rangle = \int \prod_{a=1}^{n-r} dq^a dp^a \exp\left[i \int_{t_\alpha}^{t'_\alpha} \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dt_\alpha \right]. \quad (14)$$

One should notice that the integral (14) is an integration over the canonical phase space coordinates q_a, p_a.

3 Path integral quantization of the massive spin one particle

The best example of a theory exhibiting gauge invariance is Maxwell electromagnetic. the following Lagrangian describe the theory

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \mu, \nu = 0, 1, 2, 3\,, \quad (15)$$

with the field strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, and the Abelian gauge filed A^μ which transforms under gauge transformation as $\delta A_\mu(x) = \partial_\mu \Lambda(x)$. where Λ is some arbitrary function of coordinates. Therefore, the field strength tensor $F_{\mu\nu}$ is a gauge invariant.
Schwinger [30] proved that the gauge invariant and mass can coexist. He showed, in the case of the gauge field is coupled to a conserved current J^μ and the theory is such that current correlator $\langle vac|J_\mu J_\nu|vac\rangle$ has a pole at $P^2 = 0^1$, the gauge bosons in the theory are invariably massive.

Therefore, the Lagrangian of a massive spin one particle is described as

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu - A_\mu J^\mu, \quad \mu, \nu = 0, 1, 2, 3, \quad (16)$$

This Lagrangian function is singular, since the rank of the Hess matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}. \quad (17)$$

is three.
The momenta conjugated to the fields A^μ are defined as

$$\pi^\mu = \frac{\partial l}{\partial \dot{A}_\mu} = F^{\mu 0}. \tag{18}$$

Therefore

$$\pi^0 = 0 = -H. \tag{19}$$

The spatial components of π^μ are just the electric field:

$$\pi^i = F^{i0} = E^i, \qquad i = 1, 2, 3. \tag{20}$$

Using (18), the velocities $\dot{A}_i$ can be expressed in terms of the momenta π_i as

$$\dot{A}_i = -\pi_i + \partial_i A_0. \tag{21}$$

Therefore, the Hamiltonian density is obtained as

$$H_0 = \frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} \pi^i \pi_i + \pi^i \partial_i A_0 - \frac{1}{2} m^2 (A_0 A^0 + A_i A^i) + A_0 J^0 + A_i J^i. \tag{22}$$

The set of Hamilton-Jacobi partial differential equations reads

$$H_0' = p_0 + H_0, \tag{23}$$

$$H' = \pi_0 + H = \pi_0 = 0. \tag{24}$$

Therefore, the total differential equations for the characteristic (9), (10) and (11) are

$$\begin{aligned} dA^i &= \frac{\partial H_0'}{\partial \pi_i}\, dt + \frac{\partial H'}{\partial \pi_i}\, dA^0, \\ &= -(\pi^i + \partial_i A_0)\, dt, \end{aligned} \tag{25}$$

$$\begin{aligned} d\pi^i &= -\frac{\partial H_0'}{\partial A_i}\, dt - \frac{\partial H'}{\partial A_i}\, dA^0, \\ &= (\partial_l F^{li} + m^2 A^i - J^i)\, dt, \quad l = 1, 2, 3, \end{aligned} \tag{26}$$

$$d\pi^0 = -\frac{\partial H_0'}{\partial A_0}\, dt - \frac{\partial H'}{\partial A_0}\, dA^0,$$
$$= (\partial_i \pi^i + m^2 A^0 - J^0)\, dt, \tag{27}$$

$$dp^0 = -\frac{\partial H_0'}{\partial t}\, dt - \frac{\partial H'}{\partial t}\, dA^0,$$
$$= -(A^0 \frac{\partial J_0}{\partial t} + A^i \frac{\partial J_i}{\partial t})\, dt. \tag{28}$$

To check whether the set of Eqs. (25), (26), (27) and (28) is integrable or not, let us consider the variation of Eqs. (23) and (24). In fact

$$dH_0' = F_1\, dA^0, \tag{29}$$

where

$$F_1 = (\partial_i \pi^i + m^2 A_0 - J_0) \equiv 0. \tag{30}$$

Since (29) is not identically zero, we consider it as a new constraint F_1, and one should consider the variation of F_1 too. The variation is simply zero, and it is equal to F_1.
Imposing a gain the variation of F_1

$$dF_1 = d(\partial_i \pi^i + m^2 A_0 - J_0) = \partial_i d\pi^i + m^2 dA_0. \tag{31}$$

This equation is not identically zero since

$$\partial_i \pi^i = \partial_i(-\frac{\partial H_0'}{\partial A_i} dt - \frac{\partial H'}{\partial A_i} dA^0),$$
$$= \partial_i (m^2 A^i - J^i) dt. \tag{32}$$

Therefore, we have another new constraint of the form

$$F_2 \equiv -\partial_i (J^i - m^2 A^i) = 0, \tag{33}$$

which represents the continuity equation. Finally, the variation of F_2 is identically zero since

$$dF_2 = -d[\partial_i (J^i - m^2 A^i)] \equiv 0. \tag{34}$$

Thus, no further constraints arise.
The set of Eqs. (25), (26), (27) and (28) is integrable and the equivalent partial differential equations of them are just ordinary differential equations which are take the form

$$\dot{A}^i = -\pi^i - \partial_i A_0, \tag{35}$$

$$\dot{\pi}^i = \partial_l F^{li} + m^2 A^i - J^i, \tag{36}$$

$$\dot{\pi}^0 = \partial_i \pi^i + m^2 A^0 - J^0. \tag{37}$$

One may write Eqs. (35 - 37) in a compact notation for vectors as follows:

$$\frac{\partial \vec{A}}{\partial t} + \vec{E} = -\vec{\nabla}\phi, \tag{38}$$

$$\frac{\partial \vec{E}}{\partial t} = (\vec{\nabla} \times \vec{B}) + m^2\phi - \vec{J}, \tag{39}$$

$$\vec{\nabla} \cdot \vec{E} = -m^2\phi + \rho, \tag{40}$$

where $\partial_l F^{li} = (\vec{\nabla} \times \vec{B})_i$.
Taking the curl of Eq. (38) yields

$$\frac{\partial \vec{B}}{\partial t} + (\vec{\nabla} \times \vec{E}) = 0, \tag{41}$$

where $\vec{B} = (\vec{\nabla} \times \vec{A})$, which gives

$$\vec{\nabla} \cdot \vec{B} = 0. \tag{42}$$

Equations (39) and (40) are equations for the electromagnetic field, whereas Eqs. (41) and (42) are Maxwell's equations.
Eqs. (12) and (22 - 24) lead us to the canonical action integral as

$$Z = \int d^4x \Big(-\frac{1}{4} F^{ij} F_{ij} + \frac{1}{2} \pi^i \pi_i + \pi_i \, \partial_i A_0 + \pi_i \, \dot{A}^i + \frac{1}{2} m^2 (A_0 A^0 + A_i A^i) - A_0 J^0 - A_i J^i \Big). \tag{43}$$

Making use of Eqs. (14) and (43), we obtain the path integral as

$$\langle out|S|In\rangle = \int \prod DA^i D\pi^i exp\Big[i\{\int d^4x\Big(\frac{1}{2}\,\pi^2 + \pi_i(\partial_i A_0 + \dot{A}^i) + \frac{1}{2}\,m^2(A_0A^0 + A_iA^i) - \frac{1}{4}\,F^{ij}F_{ij} - A_0J^0 - A_iJ^i\Big)\}\Big]. \quad (44)$$

Integration over π_i gives

$$\langle out|S|In\rangle = N\int \prod DA^i exp\Big[i\Big\{\int d^4x\Big(\frac{1}{2}(\partial_i A^0 + \dot{A}^i)^2 - \frac{1}{4}\,F^{ij}F_{ij} + \frac{1}{2}\,m^2(A_0A^0 + A_iA^i) - A_0J^0 - A_iJ^i\Big)\Big\}\Big],$$

$$= N\int \prod DA^i exp\Big[i\Big\{\int d^4x\Big(\frac{1}{2}\,\dot{A}^2 - \frac{1}{4}\;F^{ij}F_{ij} + J\cdot A + \frac{1}{2}\,m^2(A_0A^0 + A_iA^i) - \frac{1}{2}\,A^0(2\,\nabla\cdot\dot{A} + 2\,J_0 + \nabla^2 A_0)\Big)\Big\}\Big], \quad (45)$$

where N is a constant.
Equation (45) can be written as

$$\langle out|S|In\rangle = N\int \prod DA^i exp\Big[i\Big\{\int d^4x\Big(\frac{1}{2}\,\dot{A}^2 - \frac{1}{4}\;F^{ij}F_{ij} + J\cdot A + \frac{1}{2}\,m^2(\phi^2 + A_iA^i) - \frac{1}{2}\,\phi\,(2\,\nabla\cdot\dot{A} + 2\,\rho + \nabla^2\phi)\Big)\Big\}\Big], \quad (46)$$

Equations (38) and (40) lead us to obtain

$$\nabla\cdot\dot{A} = -\nabla^2\,\phi + m^2\phi - \rho\,. \quad (47)$$

Inserting Eq. (47) in the integral (46) gives

$$\langle out|S|In\rangle = N\int \prod DA^i exp\Big[i\Big\{\int d^4x\Big(\frac{1}{2}\,\dot{A}^2 - \frac{1}{4}\;F^{ij}F_{ij} + J\cdot A + \frac{1}{2}\,\phi\,\nabla^2\phi - \frac{1}{2}\,m^2(\phi^2 - A_iA^i)\Big)\Big\}\Big]. \quad (48)$$

4 Conclusion

We have obtained the path integral quantization of a massive spin one particle, which is a constrained field system, by using the canonical method. The equations of motion of this system are derived.

One should notice that the term $\frac{1}{2}\dot{A}^2 - \frac{1}{4}\ F^{ij}F_{ij} + J \cdot A$ leads to a solution for A, in which

$$A = \int G\,\eta\,d^3x. \tag{49}$$

Thus G has an inverse. Hence, the path integral (49) has no singular nature.

As was demonstrated by this example, gauge fixing is not necessary to obtain the path integral formulation for field theories if the canonical formulation is used. Since this system is integrable, H_0' and H' can be interpreted as infinitesimal generators of canonical transformations given by parameters t and A^0, respectively. In this case the path integral is obtained as an integration over the canonical phase-space coordinates A^i, π^i. Faddeev, Popov and Senjanovic treatments [3-5] need gauge-fixing conditions to obtain the path integral over the canonical variable.

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